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Multiple scattering methods in Casimir calculations

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Abstract
Multiple scattering formulations have been employed for more than 30 years as a method of studying the quantum vacuum or Casimir interactions between distinct bodies. Here we review the method in the simple context of δ-function potentials, so-called semitransparent bodies. (In the limit of strong coupling, a semitransparent boundary becomes a Dirichlet one.) After applying the method to rederive the Casimir force between two semitransparent plates and the Casimir self-stress on a semitransparent sphere, we obtain expressions for the Casimir energies between disjoint parallel semitransparent cylinders and between disjoint semitransparent spheres. Simplifications occur for weak and strong coupling. In particular, after performing a power series expansion in the ratio of the radii of the objects to the separations between their centers, we are able to sum the weak-coupling expansions exactly to obtain explicit closed forms for the Casimir interaction energy. The same can be done for the interaction of a weak-coupling sphere or cylinder with a Dirichlet plane. We show that the proximity force approximation (PFA), which becomes the proximity force theorem when the objects are nearly touching each other, is very poor for finite separations.

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1. Introduction

Recently, there has been a flurry of papers concerning 'exact' methods of calculating Casimir energies or forces between arbitrary distinct bodies. A most notable one is the recent paper by Emig, Graham, Jaffe, and Kardar [1]. (Details, applied to a scalar field, are supplied in [2], see also [3, 4].) Precursors include an early paper of Renne [5], rederiving Lifshitz' formula [6] in this way, the famous papers of Balian and Duplantier [7–9], work of Kenneth and Klich [10] based on the Lippmann-Schwinger formulation of scattering theory [11], papers by Bulgac, Marierski, and Wirzba [12–14] who use the modified Krein formula [15], and

[1] http://www.nhn.ou.edu/%7Emilton.

by Bordag [16, 17] who derives his results from a path integral formulation. Dalvit *et al* [18, 19] use the argument principle to calculate the interaction between conducting cylinders with parallel axes. See also Reynaud *et al* [20] and references therein.

In fact, Emig and earlier collaborators [21–23] have published a series of papers, using closely related methods to calculate numerically forces between distinct bodies, starting from periodically deformed ones. Strong deviation from the proximity force approximation (PFA) is seen, when the distance between the bodies is large compared to their radii of curvature. Bordag [16, 24] has precisely quantified the first correction to the PFA both for a cylinder and a sphere near a plane. As Gies and Klingmüller note [25], 1% deviation from the PFA occurs when the ratio of the distance between the cylinder and plate to the radius of the cylinder exceeds 0.01. We will not discuss the worldline method of Gies and collaborators [26–28] further, as that method lies rather outside our discussion here. Similar remarks apply for the work of Capasso *et al* [29], who calculate forces from stress tensors using the familiar construction of the stress tensor in terms of Green's dyadics [30, 31]. They use a numerical finite-difference engineering method.

It is clear, then, with the exception of the last two methods, that these approaches are fundamentally equivalent. We refer to all of the former methods as multiple scattering techniques. We now proceed to state the formulation in a simple, straightforward manner and apply it to various situations, all characterized by δ-function potentials. (A preliminary version of some of our results has already appeared in [32].)

2. Formalism

We begin by noting that the multiple-scattering formalism may be derived from the general formula for Casimir energies (for simplicity here we restrict our attention to a massless scalar field) [33]

$$E = \frac{\mathrm{i}}{2\tau}\,\mathrm{Tr}\ln G, \tag{2.1}$$

where τ is the 'infinite' time that the configuration exists, and G is the Green's function in the presence of a potential V satisfying (matrix notation)

$$(-\partial^2 + V)G = 1, \tag{2.2}$$

subject to some boundary conditions at infinity. (For example, we can use causal or Feynman boundary conditions, or alternatively, retarded Green's functions.) In appendix A we give a heuristic derivation of this fundamental formula.

The above formula for the Casimir energy is defined up to an infinite constant, which can be compensated at least partially by inserting a factor as Kenneth and Klich [10] do:

$$E = \frac{\mathrm{i}}{2\tau}\,\mathrm{Tr}\ln GG_0^{-1}. \tag{2.3}$$

Here G_0 satisfies, with the same boundary conditions as G, the free equation

$$-\partial^2 G_0 = 1. \tag{2.4}$$

Now we define the T-matrix (note that our definition of T differs by a factor of 2 from that in [1])

$$T = S - 1 = V(1 + G_0 V)^{-1}. \tag{2.5}$$

We then follow standard scattering theory [11], as reviewed by Kenneth and Klich in [10]. (Note that there seem to be some sign and ordering errors in that reference.) The Green's function can be written alternatively as

$$G = G_0 - G_0 T G_0 = \frac{1}{1 + G_0 V} G_0 = V^{-1} T G_0, \tag{2.6}$$

which results in two formulae for the Casimir energy

$$E = \frac{\mathrm{i}}{2\tau}\mathrm{Tr}\ln\frac{1}{1+G_0V} \tag{2.7a}$$

$$= \frac{\mathrm{i}}{2\tau}\mathrm{Tr}\ln V^{-1}T. \tag{2.7b}$$

If the potential has two disjoint parts,

$$V = V_1 + V_2, \tag{2.8}$$

it is easy to show that

$$T = (V_1+V_2)(1-G_0T_1)(1-G_0T_1G_0T_2)^{-1}(1-G_0T_2), \tag{2.9}$$

where

$$T_i = V_i(1+G_0V_i)^{-1}, \qquad i = 1, 2. \tag{2.10}$$

Thus, we can write the general expression for the interaction between the two bodies (potentials) in two alternative forms:

$$E_{12} = -\frac{\mathrm{i}}{2\tau}\,\mathrm{Tr}\ln(1-G_0T_1G_0T_2) \tag{2.11a}$$

$$= -\frac{\mathrm{i}}{2\tau}\,\mathrm{Tr}\ln(1-V_1G_1V_2G_2), \tag{2.11b}$$

where

$$G_i = (1+G_0V_i)^{-1}G_0, \qquad i = 1, 2. \tag{2.12}$$

The first form is exactly that given by Emig *et al* in [1], and by Kenneth and Klich in [10], while the latter is actually easy to use if we know the individual Green's functions. (The effort involved in calculation using either of the two is same.) In fact, the general form (2.11*a*) was recognized earlier and was applied to planar geometries by Maia Neto, Lambrecht, and Reynaud in [20, 34, 35]. In fact, Renne [5] essentially used equation (2.11*b*) to derive the Lifshitz formula in 1971.

3. Casimir interaction between δ-plates

We now use the above-mentioned second formula (2.11*b*) to calculate the Casimir energy between two parallel semitransparent plates, with potential

$$V = \lambda_1\delta(z-z_1) + \lambda_2\delta(z-z_2), \tag{3.1}$$

where the dimension of λ_i is L^{-1}. The free reduced Green's function is (where we have performed the evident Fourier transforms in time and the transverse directions)

$$g_0(z,z') = \frac{1}{2\kappa}\mathrm{e}^{-\kappa|z-z'|}, \qquad \kappa^2 = \zeta^2 + k^2. \tag{3.2}$$

Here $\mathbf{k} = \mathbf{k}_\perp$ is the transverse momentum, and $\zeta = -\mathrm{i}\omega$ is the Euclidean frequency. The Green's function associated with a single δ-function potential is

$$g_i(z,z') = \frac{1}{2\kappa}\left(\mathrm{e}^{-\kappa|z-z'|} - \frac{\lambda_i}{\lambda_i+2\kappa}\,\mathrm{e}^{-\kappa|z-z_i|}\,\mathrm{e}^{-\kappa|z'-z_i|}\right). \tag{3.3}$$

Then the energy per unit area is

$$\mathcal{E} = \frac{1}{16\pi^3}\int \mathrm{d}\zeta \int \mathrm{d}^2k \int \mathrm{d}z\,\ln(1-A)(z,z), \tag{3.4}$$

where, in virtue of the δ-function potentials ($a = |z_2 - z_1|$)

$$A(z,z') = \frac{\lambda_1\lambda_2}{4\kappa^2}\delta(z-z_1)\left(1-\frac{\lambda_1}{\lambda_1+2\kappa}\right)\mathrm{e}^{-\kappa|z_1-z_2|}\left(1-\frac{\lambda_2}{\lambda_2+2\kappa}\right)\mathrm{e}^{-\kappa|z'-z_2|}$$
$$= \frac{\lambda_1}{\lambda_1+2\kappa}\frac{\lambda_2}{\lambda_2+2\kappa}\mathrm{e}^{-\kappa a}\,\mathrm{e}^{-\kappa|z'-z_2|}\delta(z-z_1). \tag{3.5}$$

We expand the logarithm according to

$$\ln(1-A) = -\sum_{s=1}^{\infty}\frac{A^s}{s}. \tag{3.6}$$

For example, the leading term is easily seen to be

$$\mathcal{E}^{(2)} = -\frac{\lambda_1\lambda_2}{16\pi^3}\int\frac{\mathrm{d}\zeta\,\mathrm{d}^2k}{4\kappa^2}\mathrm{e}^{-2\kappa a} = -\frac{\lambda_1\lambda_2}{32\pi^2 a}, \tag{3.7}$$

which uses the transforms in polar coordinates,

$$\mathrm{d}\zeta\,\mathrm{d}^2k = \mathrm{d}\kappa\,\kappa^2\,\mathrm{d}\Omega. \tag{3.8}$$

In general, it is easy to check that, because $A(z,z')$ factorizes here, $A(z,z') = B(z)C(z')$, $\mathrm{Tr}\,A^n = (\mathrm{Tr}\,A)^n$, or

$$\mathrm{Tr}\ln(1-A) = \ln(1-\mathrm{Tr}\,A), \tag{3.9}$$

so the Casimir interaction between the two semitransparent plates is

$$\mathcal{E} = \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}\kappa\,\kappa^2\ln\left(1-\frac{\lambda_1}{\lambda_1+2\kappa}\mathrm{e}^{-\kappa a}\frac{\lambda_2}{\lambda_2+2\kappa}\mathrm{e}^{-\kappa a}\right), \tag{3.10}$$

which is exactly the well-known result, for example given in [36].

4. Casimir self-energy for a single semitransparent sphere

Before we embark on new calculations, let us also confirm the known result for the self-stress on a single sphere of radius a using this formalism. (This demonstrates, as did the rederivation of the Boyer result [37] by Balian and Duplantier in [8], that the multiple scattering method is equally applicable for the calculation of self-energies.) We start from the general formula (2.7*a*), where

$$V(\mathbf{r},\mathbf{r}') = \lambda\delta(r-a)\delta(\mathbf{r}-\mathbf{r}'). \tag{4.1}$$

We use the Fourier representation for the propagator in Euclidean space,

$$G_0(\mathbf{r},\mathbf{r}') = \frac{\mathrm{e}^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2+\zeta^2}, \tag{4.2}$$

as well as the partial wave expansion of the plane wave

$$\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = \sum_{lm}4\pi\mathrm{i}^l j_l(kr)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{k}}). \tag{4.3}$$

Then, from the orthonormality of the spherical harmonics,

$$\int\mathrm{d}\hat{\mathbf{k}}\,Y^*_{lm}(\hat{\mathbf{k}})Y_{l'm'}(\hat{\mathbf{k}}) = \delta_{ll'}\delta_{mm'}, \tag{4.4}$$

we obtain the representation

$$G_0(\mathbf{r},\mathbf{r}') = \frac{2}{\pi}\sum_{lm}\int_0^\infty\frac{\mathrm{d}k\,k^2}{k^2+\zeta^2}j_l(kr)j_l(kr')Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}'). \tag{4.5}$$

Now we combine the representation of the free Green's function with the spherical potential (4.1) to obtain

$$(G_0V)(\mathbf{r},\mathbf{r}') = \frac{2\lambda}{\pi}\delta(r'-a)\sum_{lm}\int_0^\infty \frac{\mathrm{d}k\,k^2}{k^2+\zeta^2} j_l(ka)j_l(kr)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}'). \tag{4.6}$$

When this, or powers of this, is traced (that is, $\mathbf{r}$ and $\mathbf{r}'$ are set equal, and integrated over), we obtain a poorly defined expression; to regulate this, we assume $r \neq a$, for example, $r < a$. (This is a type of point-split regulation.) Then, because

$$j_l(ka) = \tfrac{1}{2}\big(h_l^{(1)}(ka) + h_l^{(2)}(ka)\big) = \tfrac{1}{2}\big(h_l^{(1)}(ka) + (-1)^l h_l^{(1)}(-ka)\big), \tag{4.7}$$

while $j_l(kr) = (-1)^l j_l(-kr)$, we see that the k integration in equation (4.6) can be evaluated as[2]

$$\int_0^\infty \frac{\mathrm{d}k\,k^2}{k^2+\zeta^2} j_l(ka)j_l(kr) = \frac{\pi}{a}K_{l+1/2}(|\zeta|a)I_{l+1/2}(|\zeta|r), \qquad r < a. \tag{4.8}$$

Thus, it is easily seen that an arbitrary power of G_0V has trace

$$\mathrm{Tr}(G_0V)^n = (\lambda a)^n \sum_{lm}(K_{l+1/2}(|\zeta|a)I_{l+1/2}(|\zeta|a))^n, \tag{4.9}$$

and therefore the total self-energy of the semitransparent sphere is given by the well-known expression [38, 39]

$$E = \frac{1}{2\pi a}\sum_{l=0}^\infty (2l+1)\int_0^\infty \mathrm{d}x\,\ln\big(1+\lambda a I_{l+1/2}(x)K_{l+1/2}(x)\big), \qquad x = |\zeta|a. \tag{4.10}$$

Actually, a slightly different form involving integration by parts was given in [40, 41], which results in the energy being finite through order λ^2. In order λ^3 there is a divergence which is associated with surface energy [42].

5. 2 + 1 spatial geometries

We now proceed to apply this method to the interaction between bodies, which leads, as Emig *et al* point out in [1, 2], to a multipole expansion. In this section we illustrate this idea with a 2 + 1 dimensional version, which allows us to describe, for example, cylinders with parallel axes. We seek an expansion of the free Green's function for $\mathbf{R} = \mathbf{R}_\perp$ entirely in the x–y plane,

$$G_0(\mathbf{R}+\mathbf{r}'-\mathbf{r}) = \frac{\mathrm{e}^{\mathrm{i}|\omega||\mathbf{r}-\mathbf{R}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{R}-\mathbf{r}'|} = \int\frac{\mathrm{d}k_z}{2\pi}\,\mathrm{e}^{\mathrm{i}k_z(z-z')}g_0(\mathbf{r}_\perp-\mathbf{R}_\perp-\mathbf{r}'_\perp), \tag{5.1}$$

where the reduced Green's function is

$$g_0(\mathbf{r}_\perp-\mathbf{R}_\perp-\mathbf{r}'_\perp) = \int\frac{(\mathrm{d}^2k_\perp)}{(2\pi)^2}\frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}_\perp\cdot\mathbf{R}_\perp}\,\mathrm{e}^{\mathrm{i}\mathbf{k}_\perp\cdot(\mathbf{r}_\perp-\mathbf{r}'_\perp)}}{k_\perp^2+k_z^2+\zeta^2}. \tag{5.2}$$

As long as the two potentials do not overlap, so that we have $\mathbf{r}_\perp-\mathbf{R}_\perp-\mathbf{r}'_\perp \neq 0$, we can write an expansion in terms of modified Bessel functions:

$$g_0(\mathbf{r}_\perp-\mathbf{R}_\perp-\mathbf{r}'_\perp) = \sum_{m,m'} I_m(\kappa r)\,\mathrm{e}^{\mathrm{i}m\phi}I_{m'}(\kappa r')\mathrm{e}^{-\mathrm{i}m'\phi'}\tilde{g}^0_{m,m'}(\kappa R), \qquad \kappa^2 = k_z^2+\zeta^2. \tag{5.3}$$

[2] Of course, this result is the immediate consequence of the usual partial wave expansion

$$G_0(\mathbf{r},\mathbf{r}') = ik\sum_{lm} j_l(kr_<)h_l^{(1)}(kr_>)Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}'), \qquad k = |\omega|.$$

The point of our slightly more elaborate approach here is that it generalizes to the corresponding two-body case, see equation (5.7).

By performing a Fourier transform, and using the definition of the Bessel function

$$i^m J_m(kr) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-im\phi} e^{ikr\cos\phi}, \tag{5.4}$$

we easily find

$$\tilde{g}^0_{m,m'}(\kappa R) = \frac{1}{2\pi}\int_0^\infty \frac{dk\, k}{k^2+\kappa^2} J_{m-m'}(kR)\frac{J_m(kr)J_{m'}(kr')}{I_m(\kappa r)I_{m'}(\kappa r')}, \tag{5.5}$$

which is in fact independent of r, r'.

As in the previous section, the k integral here can actually be evaluated as a contour integral, as Bordag noted in [16]. No point-splitting is required here, because the bodies are non-overlapping, so $r/R, r'/R < 1$. We write the dominant Bessel function in terms of Hankel functions,

$$J_{m-m'}(x) = \frac{1}{2}\left[H^{(1)}_{m-m'}(x) + H^{(2)}_{m-m'}(x)\right] = \frac{1}{2}\left[H^{(1)}_{m-m'}(x) + (-1)^{m-m'+1}H^{(1)}_{m-m'}(-x)\right], \tag{5.6}$$

and then we can carry out the integral over k by closing the contour in the upper half plane. We are left with

$$\int_0^\infty \frac{dx\, x}{x^2+y^2} J_{m-m'}(x)J_m(xr/R)J_{m'}(xr'/R) = (-1)^{m'}K_{m-m'}(y)I_m(yr/R)I_{m'}(yr'/R), \tag{5.7}$$

and therefore the reduced Green's function has the simple form

$$\tilde{g}^0_{m,m'}(\kappa R) = \frac{(-1)^{m'}}{2\pi}K_{m-m'}(\kappa R). \tag{5.8}$$

Thus we can derive an expression for the interaction energy per unit length between the two bodies, in terms of discrete matrices,

$$\mathfrak{E} \equiv \frac{E_{\text{int}}}{L} = \frac{1}{8\pi^2}\int d\zeta\, dk_z \ln\det(1 - \tilde{g}^0 t_1 \tilde{g}^{0\top} t_2), \tag{5.9}$$

where $\top$ denotes transpose, and where the T matrix elements are given by

$$t_{mm'} = \int dr\, r\, d\phi \int dr'\, r'\, d\phi'\, I_m(\kappa r)\, e^{-im\phi} I_{m'}(\kappa r')\, e^{im'\phi'} T(r,\phi;r',\phi'). \tag{5.10}$$

5.1. Interaction between semitransparent cylinders

Consider, as an example, two parallel semitransparent cylinders, of radii a and b, respectively, lying outside each other and described by the potentials

$$V_1 = \lambda_1\delta(r-a), \qquad V_2 = \lambda_2\delta(r'-b), \tag{5.11}$$

with the separation between the centers R satisfying $R > a+b$. It is easy to work out the scattering matrix in this situation,

$$T_1 = V_1 - V_1G_0V_1 + V_1G_0V_1G_0V_1 - \cdots, \tag{5.12}$$

so the matrix element is easily seen to be

$$(t_1)_{mm'} = 2\pi\lambda_1 a\delta_{mm'}\frac{I_m^2(\kappa a)}{1+\lambda_1 a I_m(\kappa a)K_m(\kappa a)}. \tag{5.13}$$

Again, we used here the regularized integral[3]

$$\int_0^\infty \frac{dk\, k}{k^2+\kappa^2}J_m(kr)J_m(kr') = K_m(\kappa r')I_m(\kappa r), \qquad r < r'. \tag{5.14}$$

[3] Again, this is equivalent to the use of the two-dimensional Green's function

$$H_0(kP) = \sum_{m=-\infty}^{\infty} i^m\, e^{-im\phi'} J_m(k\rho')\, e^{im\phi} H^{(1)}_m(k\rho), \qquad \rho' < \rho,$$

where $P = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}$.

Thus the Casimir energy per unit length is

$$\mathfrak{E} = \frac{1}{4\pi}\int_0^\infty \mathrm{d}\kappa\,\kappa\,\mathrm{tr}\ln(1-A), \tag{5.15}$$

where

$$A = B(a)B(b), \tag{5.16}$$

in terms of the matrices

$$B_{mm'}(a) = K_{m+m'}(\kappa R)\frac{\lambda_1 a I_{m'}^2(\kappa a)}{1+\lambda_1 a I_{m'}(\kappa a)K_{m'}(\kappa a)}. \tag{5.17}$$

5.2. *Interaction between cylinder and plane*

As a check, let us rederive the result derived by Bordag in [16] for a cylinder in front of a Dirichlet plane perpendicular to the x-axis. We start from the interaction (2.11a) written in terms of $\bar{G}_2$, the deviation from the free Green's function induced by a single potential,

$$\bar{G}_2 = G_2 - G_0 = -G_0 T_2 G_0, \tag{5.18}$$

so the interaction energy has the form

$$E = -\frac{\mathrm{i}}{2\tau}\,\mathrm{Tr}\ln(1+T_1\bar{G}_2). \tag{5.19}$$

When the second body is a Dirichlet plane, $\bar{G}$ may be found by the method of images, with the origin taken at the center of the cylinder,

$$\bar{G}(\mathbf{r},\mathbf{r}') = -G_0(\mathbf{r},\bar{\mathbf{r}}'), \qquad \bar{\mathbf{r}}' = (R-x', y', z'), \tag{5.20}$$

where R is the distance between the center of the cylinder and its image at $\mathbf{R}_\perp$, that is, $R/2$ is the distance between the center of the cylinder and the plane. (We keep R here, rather than $R/2 = D$, because of the close connection to the two cylinder case.) Now we encounter the two-dimensional Green's function

$$g(\mathbf{r}_\perp + \mathbf{r}'_\perp - \mathbf{R}_\perp) = \sum_{mm'} I_m(\kappa r)I_{m'}(\kappa r')\,\mathrm{e}^{\mathrm{i}m\phi}\,\mathrm{e}^{\mathrm{i}m'\phi'} g_{mm'}(\kappa R), \tag{5.21}$$

(because the cylinder has $y \to -y$ reflection symmetry) where the argument given above yields

$$g_{mm'}(\kappa R) = \frac{1}{2\pi}K_{m+m'}(\kappa R). \tag{5.22}$$

Thus the interaction between the semitransparent cylinder and a Dirichlet plane is

$$\mathfrak{E} = \frac{1}{4\pi}\int_0^\infty \kappa\,\mathrm{d}\kappa\,\mathrm{tr}\ln(1-B(a)), \tag{5.23}$$

where $B(a)$ is given by equation (5.17). In the strong-coupling limit this result agrees with that given by Bordag, because

$$\mathrm{tr}\,B^s = \mathrm{tr}\,\tilde{B}^s, \qquad \tilde{B}_{mm'} = \frac{1}{K_m(\kappa a)}K_{m+m'}(\kappa R)I_{m'}(\kappa a). \tag{5.24}$$

5.3. Weak-coupling

In weak coupling, the formula (5.15) for the interaction energy between two cylinders is

$$\mathfrak{E} = -\frac{\lambda_1\lambda_2 ab}{4\pi R^2}\sum_{m,m'=-\infty}^{\infty}\int_0^\infty \mathrm{d}x\, x K_{m+m'}^2(x) I_m^2(xa/R) I_{m'}^2(xb/R). \tag{5.25}$$

Similarly, the energy of interaction between a weakly-coupled cylinder and a Dirichlet plane is from equation (5.23)

$$\mathfrak{E} = -\frac{\lambda a}{4\pi R^2}\sum_{m=-\infty}^{\infty}\int_0^\infty \mathrm{d}x\, x K_{2m}(x) I_m^2(xa/R). \tag{5.26}$$

5.4. Power series expansion

It is straightforward to develop a power series expansion for the interaction between weakly-coupled semitransparent cylinders. One merely exploits the small argument expansion for the modified Bessel functions $I_m(xa/R)$ and $I_{m'}(xb/R)$:

$$I_m^2(x) = \left(\frac{x}{2}\right)^{2|m|}\sum_{n=0}^{\infty} Z_{|m|,n}\left(\frac{x}{2}\right)^{2n}, \tag{5.27}$$

where the coefficients $Z_{m,n}$ are

$$\begin{aligned} Z_{m,n} &= \sum_{k=0}^{n}\frac{1}{k!(n-k)!\Gamma(k+m+1)\Gamma(n-k+m+1)} \\ &= \frac{2^{2(m+n)}\Gamma\left(m+n+\frac{1}{2}\right)}{\sqrt{\pi}n!(2m+n)!\Gamma(m+n+1)}. \end{aligned} \tag{5.28}$$

The Casimir energy per unit length (5.25) is now given as

$$\begin{aligned} \mathfrak{E} = -\frac{\lambda_1 a\lambda_2 b}{4\pi R^2}\int_0^\infty \mathrm{d}x\, x \sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\left(\frac{xa}{2R}\right)^{2|m|} \\ \times Z_{|m|,n}\left(\frac{xa}{2R}\right)^{2n}\left(\frac{xb}{2R}\right)^{2|m'|} Z_{|m'|,n'}\left(\frac{xb}{2R}\right)^{2n'} K_{m+m'}^2(x). \end{aligned} \tag{5.29}$$

Reordering terms gives a more compact formula

$$\begin{aligned} \mathfrak{E} = -\frac{\lambda_1 a\lambda_2 b}{4\pi R^2}\sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} Z_{|m|,n}\left(\frac{a}{R}\right)^{2(|m|+n)} \\ \times Z_{|m'|,n'}\left(\frac{b}{R}\right)^{2(|m'|+n')} J_{|m|+|m'|+n+n',m+m'}, \end{aligned} \tag{5.30}$$

where the two index symbol $J_{p,q}$ represents the integral over x, which evaluates to

$$J_{p,q} = 2\int_0^\infty \mathrm{d}x\left(\frac{x}{2}\right)^{2p+1} K_q^2(x) = \frac{\sqrt{\pi}p!\Gamma(p+q+1)\Gamma(p-q+1)}{2^{2p+2}\Gamma\left(p+\frac{3}{2}\right)}. \tag{5.31}$$

In order to simplify the power series expansion in terms of $\frac{a}{R}$ and $\frac{b}{R}$ we need to reorder the m-sums so that only non-negative values of m appear. There are several ways to break up the m sums; one of them is to decompose the sum into the $m = m' = 0$ term, the m, m' same-sign terms, and the m, m' different-sign terms, giving

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2}\left[\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} Z_{0,n}\left(\frac{a}{R}\right)^{2n} Z_{0,n'}\left(\frac{b}{R}\right)^{2n'} J_{n+n',0}\right.$$
$$+ 2\sum_{m=1}^{\infty}\sum_{m'=0}^{\infty}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} Z_{m,n}\left(\frac{a}{R}\right)^{2(m+n)} Z_{m',n'}\left(\frac{b}{R}\right)^{2(m'+n')} J_{m+m'+n+n',m+m'}$$
$$\left.+ 2\sum_{m=0}^{\infty}\sum_{m'=1}^{\infty}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} Z_{m,n}\left(\frac{a}{R}\right)^{2(m+n)} Z_{m',n'}\left(\frac{b}{R}\right)^{2(m'+n')} J_{m+m'+n+n',m-m'}\right]. \tag{5.32}$$

It is now possible to combine the multiple infinite power series into a single infinite power series, where each term is given by (possible multiple) finite sum(s). In this case we get an amazingly simple result

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{4\pi R^2}\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{a}{R}\right)^{2n} P_n(\mu), \tag{5.33}$$

where $\mu = b/a$, and where by inspection we identify the binomial coefficients

$$P_n(\mu) = \sum_{k=0}^{n}\binom{n}{k}^2 \mu^{2k}. \tag{5.34}$$

Remarkably, it is possible to perform the sums [43], so we obtain the following closed form for the interaction between two weakly-coupled cylinders:

$$\mathfrak{E} = -\frac{\lambda_1 a \lambda_2 b}{8\pi R^2}\left[\left(1 - \left(\frac{a+b}{R}\right)^2\right)\left(1 - \left(\frac{a-b}{R}\right)^2\right)\right]^{-1/2}. \tag{5.35}$$

We note that in the limit $R - a - b = d \to 0$, d being the distance between the closest points on the two cylinders, we recover the proximity force theorem in this case (B.4),

$$U(d) = -\frac{\lambda_1\lambda_2}{32\pi}\sqrt{\frac{2ab}{R}}\frac{1}{d^{1/2}}, \qquad d \ll a, b. \tag{5.36}$$

In figures 1–2 we compare the exact energy (5.35) with the proximity force approximation (5.36). Evidently, the former approaches latter when the sum of the radii $a+b$ of the cylinders approaches the distance R between their centers. The rate of approach is linear (with slope $3/2$) for the case of equal radii, but with slope $b^2/4a^2$ when $a \ll b$. More precisely, the ratio of the exact energy to the PFA is

$$\frac{\mathfrak{E}}{U} \approx 1 - \frac{1+\mu+\mu^2}{4\mu}\frac{d}{R} \approx 1 - \frac{R^2 - aR + a^2}{4a(R-a)}\frac{d}{R}. \tag{5.37}$$

This correction to the PFA is derived by another method in appendix C. The reader should note that the PFA is actually only defined in the limit $d \to 0$, so the functional form away from that point is ambiguous. Corrections to the PFA depend upon the specific form assumed for $U(d)$.

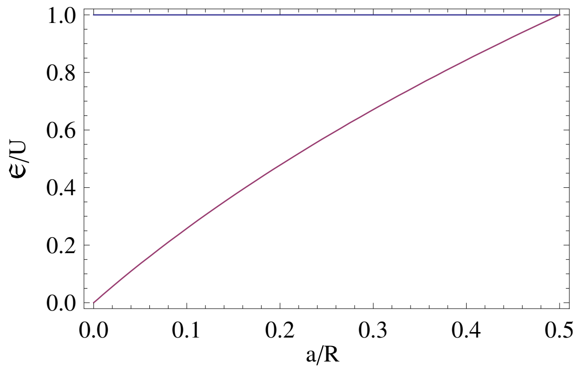

Figure 1. Plotted is the ratio of the exact interaction energy (5.35) of two weakly-coupled cylinders to the proximity force approximation (5.36) as a function of the cylinder radius a for $a = b$.

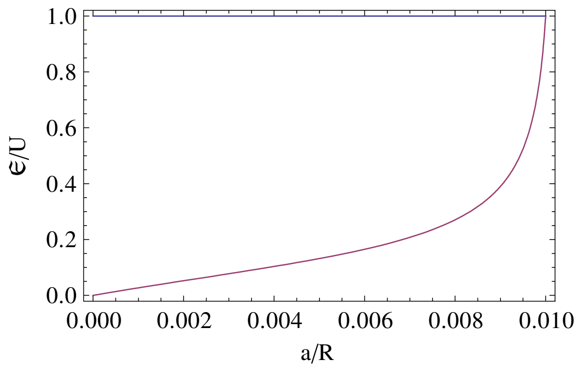

Figure 2. Plotted is the ratio of the exact interaction energy (5.35) of two weakly-coupled cylinders to the proximity force approximation (5.36) as a function of the cylinder radius a for $b/a = 99$.

5.5. Exact result for interaction between plane and cylinder

In exactly the same way, starting from equation (5.26), we can obtain a closed-form result for the interaction energy between a Dirichlet plane and a weakly-coupled cylinder of radius a separated by a distance $R/2$. The result is again quite simple:

$$\mathfrak{E} = -\frac{\lambda a}{4\pi R^2}\left[1 - \left(\frac{2a}{R}\right)^2\right]^{-3/2}. \tag{5.38}$$

In the limit as $d \to 0$, this agrees with the PFA:

$$U(d) = -\frac{\lambda}{64\pi}\frac{\sqrt{2a}}{d^{3/2}}. \tag{5.39}$$

Note again that this form is ambiguous: the proximity force theorem is equally well satisfied if we replace a by $R/2$, for example, in $U(d)$. The comparison between this PFA and the exact result (5.38) is given in figure 3. The FPA given here corresponds to that of Bordag [17] in the limit of weak coupling for the cylinder and strong coupling for the plane.

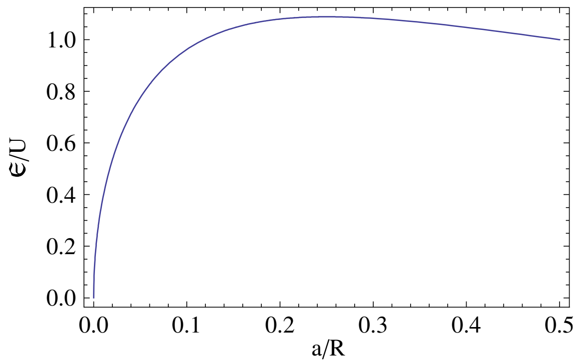

Figure 3. Plotted is the ratio of the exact interaction energy (5.38) of a weakly-coupled cylinder above a Dirichlet plane to the proximity force approximation (5.39) as a function of the cylinder radius a.

5.6. Strong coupling (Dirichlet) limit

The interaction between Dirichlet cylinders is given by equation (5.15) in the limit $\lambda_1 = \lambda_2 \to \infty$, that is

$$\mathfrak{E} = \frac{1}{4\pi R^2}\int_0^\infty \mathrm{d}x\, x\, \mathrm{tr}\ln(1 - A), \tag{5.40a}$$

where

$$A_{mm'} = \sum_{m''} K_{m+m''}(x)K_{m''+m'}(x)\frac{I_{m''}(xa/R)}{K_{m''}(xa/R)}\frac{I_{m'}(xb/R)}{K_{m'}(xb/R)}. \tag{5.40b}$$

Here the trace of the logarithm can be interpreted as in equation (3.6).

Because it no longer appears possible to obtain a closed-form solution, we want to verify analytically that as the surfaces of the two cylinders nearly touch each other, we recover the result of the proximity force theorem. We use a variation of the scheme explained by Bordag for a cylinder next to a plane in [16]. (The analysis is a bit simpler in the weak-coupling case, which leads to equation (5.36), see appendix C.) First we replace the products of Bessel functions in A by their leading uniform asymptotic approximants for all m's large:

$$B_{mm''}(a)B_{m''m'}(b) \sim \frac{1}{2\pi}\frac{1}{\sqrt{m+m''}}\frac{1}{\sqrt{m'+m''}}$$
$$\times\left(1+\left(\frac{x}{m+m''}\right)^2\right)^{-1/4}\left(1+\left(\frac{x}{m'+m''}\right)^2\right)^{-1/4}\mathrm{e}^{-\chi}, \tag{5.41}$$

where the exponent is

$$\chi = (m+m'')\eta\left(\frac{x}{m+m''}\right) + (m'+m'')\eta\left(\frac{x}{m'+m''}\right) - 2m''\eta\left(\frac{xa}{m''R}\right) - 2m'\eta\left(\frac{xb}{m'R}\right), \tag{5.42}$$

in terms of

$$\eta(z) = t^{-1} + \ln\frac{z}{1+t^{-1}}, \qquad \eta'(z) = \frac{1}{zt}, \qquad \eta''(z) = -\frac{t}{z^2}, \tag{5.43}$$

and

$$t = (1+z^2)^{-1/2}. \tag{5.44}$$

We write the trace of the sth power of A as (summed on repeated indices)

$$(A^s)_{m_1m_1} = B_{m_1m_1'}(a)B_{m_1'm_2}(b)B_{m_2m_2'}(a)B_{m_2'm_3}(b)\cdots B_{m_sm_s'}(a)B_{m_s'm_1}(b). \tag{5.45}$$

We rescale variables in terms of a large variable M and relatively small variables:

$$m_i' = M\alpha_i, \qquad m_i = M\beta_i, \tag{5.46}$$

where without loss of generality we take only $2s-1$ of the α's and β's as independent:

$$\sum_{i=1}^{s}(\alpha_i+\beta_i) = s. \tag{5.47}$$

This normalization is chosen so that at the critical point where $\chi = 0$ for $a+b = R$,

$$\alpha_i = \frac{a}{R} \qquad \beta_i = 1-\frac{a}{R}, \quad \forall i. \tag{5.48}$$

Away from this point, we consider fluctuations,

$$\alpha_i = \frac{a}{R}+\hat\alpha_i, \qquad \beta_i = 1-\frac{a}{R}+\hat\beta_i, \tag{5.49}$$

with the constraint

$$\sum_{i=1}^{s}(\hat\alpha_i+\hat\beta_i) = 0. \tag{5.50}$$

The Jacobian of this transformation is sM^{2s-1}.

Now, we expand the exponent in $\operatorname{tr} A^s$, to first order in $d = R-a-b$, and to second order in $\hat\alpha_i$, $\hat\beta_i$. The result is

$$\chi = \frac{2Msd}{tR} + Mt\left(\frac{R}{a}-1\right)\sum_{i=1}^{s}\left[\hat\alpha_i - \frac{1}{2}\frac{a}{R-a}(\hat\beta_i+\hat\beta_{i+1})\right]^2 + \frac{Mt}{4}\frac{a}{R-a}\sum_{i=1}^{s}(\hat\beta_i-\hat\beta_{i+1})^2. \tag{5.51}$$

The $\hat\alpha_i$ terms lead to trivial Gaussian integrals. The difficulty with the quadratic $\hat\beta_i$ terms is that only $s-1$ of the differences are independent. But, in view of the constraint (5.50) there are only $s-1$ independent β_i variables. In fact, it is easy to check that

$$\sum_{i=1}^{s}(\hat\beta_i-\hat\beta_{i+1})^2 = \sum_{i=1}^{s-1}\frac{i+1}{i}\left[\hat\beta_i-\hat\beta_{i+1}+\frac{1}{i+1}\sum_{j=i+1}^{s-1}(\hat\beta_j-\hat\beta_{j+1})\right]^2, \tag{5.52}$$

which now enables us to perform each successive $\hat\beta_i-\hat\beta_{i+1}$ integration. The Jacobian of the transformation to the difference variables $u_i = \hat\beta_i-\hat\beta_{i+1}$, $i = 1,\ldots,s-1$, is $1/s$. Thus, we can immediately write down

$$\begin{aligned}
\mathfrak{E} &\sim -\frac{1}{4\pi R^2}\int_0^\infty \mathrm{d}z\, z\sum_{s=1}^{\infty}\frac{t^s}{s}\int_0^\infty \mathrm{d}M\,\frac{M^{2s+1}}{(2\pi M)^s}\,\mathrm{e}^{-2Msd/tR}\\
&\quad\times\left[\int_{-\infty}^{\infty}\mathrm{d}\alpha_i\,\mathrm{e}^{-Mt(R-a)\alpha_i^2/a}\right]^s\prod_{i=i}^{s-1}\int_{-\infty}^{\infty}\mathrm{d}u_i\,\mathrm{e}^{-Mat\sum_{i=1}^{s-1}\frac{i+1}{i}u_i^2/4(R-a)}\\
&= -\frac{1}{4\pi R^2}\int_0^\infty \mathrm{d}z\, z\sum_{s=1}^{\infty}\frac{1}{s}\int_0^\infty \mathrm{d}M\,M^{2s+1}\frac{t^s}{(2\pi M)^s}\,\mathrm{e}^{-2Msd/tR}\\
&\quad\times\left[\frac{\pi a}{(R-a)Mt}\right]^{s/2}\left[\frac{4\pi(R-a)}{Mta}\right]^{(s-1)/2}s^{-1/2}\\
&= -\frac{\sqrt{2a(R-a)}\pi^3}{3840R^3}\left(\frac{R}{d}\right)^{5/2},
\end{aligned} \tag{5.53}$$

which is exactly the result expected from the proximity force theorem, according to equation (B.5).

We will forego further discussion of strong coupling, and presentation of numerical results, for these have been extensively discussed in several recent papers, especially in [2].

6. Three-dimensional formalism

The three-dimensional formalism is very similar. In this case, the free Green's function has the representation

$$G_0(\mathbf{R}+\mathbf{r}'-\mathbf{r}) = \sum_{lm,l'm'} j_l(i|\zeta|r) j_{l'}(i|\zeta|r') Y^*_{lm}(\hat{\mathbf{r}}) Y_{l'm'}(\hat{\mathbf{r}}') g_{lm,l'm'}(\mathbf{R}). \tag{6.1}$$

The reduced Green's function can be written in the form

$$g^0_{lm,l'm'}(\mathbf{R}) = (4\pi)^2 \mathrm{i}^{l'-l} \int \frac{(\mathrm{d}\mathbf{k})}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{R}}}{k^2+\zeta^2} \frac{j_l(kr) j_{l'}(kr')}{j_l(i|\zeta|r) j_{l'}(i|\zeta|r')} Y_{lm}(\hat{\mathbf{k}}) Y^*_{l'm'}(\hat{\mathbf{k}}). \tag{6.2}$$

Now we use the plane-wave expansion (4.3) once again, this time for $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{R}}$, so now we encounter something new, an integral over three spherical harmonics,

$$\int \mathrm{d}\hat{\mathbf{k}}\, Y_{lm}(\hat{\mathbf{k}}) Y^*_{l'm'}(\hat{\mathbf{k}}) Y^*_{l''m''}(\hat{\mathbf{k}}) = C_{lm,l'm',l''m''}, \tag{6.3}$$

where

$$C_{lm,l'm',l''m''} = (-1)^{m'+m''} \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}. \tag{6.4}$$

The three-j symbols (Wigner coefficients) here vanish unless $l+l'+l''$ is even. This fact is crucial, since because of it we can follow the previous method of writing $j_{l''}(kR)$ in terms of Hankel functions of the first and second kind, using the reflection property of the latter,

$$h^{(2)}_{l''}(kR) = (-1)^{l''} h^{(1)}_{l''}(-kR), \tag{6.5}$$

and then extending the k integral over the entire real axis to a contour integral closed in the upper half plane. The residue theorem then supplies the result for the reduced Green's function[4]

$$g^0_{lm,l'm'}(\mathbf{R}) = 4\pi \mathrm{i}^{l'-l} \sqrt{\frac{2|\zeta|}{\pi R}} \sum_{l''m''} C_{lm,l'm',l''m''} K_{l''+1/2}(|\zeta|R) Y_{l''m''}(\hat{\mathbf{R}}). \tag{6.6}$$

6.1. Casimir interaction between semitransparent spheres

For the case of two semitransparent spheres that are totally outside each other,

$$V_1(r) = \lambda_1 \delta(r-a), \qquad V_2(r') = \lambda_2 \delta(r'-b), \tag{6.7}$$

in terms of spherical coordinates centered on each sphere, it is again very easy to calculate the scattering matrices,

$$T_1(\mathbf{r},\mathbf{r}') = \frac{\lambda_1}{a^2} \delta(r-a)\delta(r'-a) \sum_{lm} \frac{Y_{lm}(\hat{\mathbf{r}}) Y^*_{lm}(\hat{\mathbf{r}}')}{1+\lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)}, \tag{6.8}$$

[4] This differs by a (conventional) factor of $|\zeta|$ from the quantity $U_{lml'm'}$ defined by Emig *et al* [2].

and then the harmonic transform is very similar to that seen in equation (5.13), ($k = i|\zeta|$)

$$
\begin{aligned}
(t_1)_{lm,l'm'} &= \int (\mathrm{d}\mathbf{r})(\mathrm{d}\mathbf{r}')\, j_l(kr) Y^*_{lm}(\hat{\mathbf{r}}) j_{l'}(kr') Y_{l'm'}(\hat{\mathbf{r}}') T_1(\mathbf{r},\mathbf{r}') \\
&= \delta_{ll'}\delta_{mm'}(-1)^l \frac{\lambda_1 a\pi}{2|\zeta|} \frac{I^2_{l+1/2}(|\zeta|a)}{1+\lambda_1 a K_{l+1/2}(|\zeta|a) I_{l+1/2}(|\zeta|a)}.
\end{aligned} \tag{6.9}
$$

Let us suppose that the two spheres lie along the z-axis, that is, $\mathbf{R} = R\hat{\mathbf{z}}$. Then we can simplify the expression for the energy somewhat by using $Y_{lm}(\theta = 0) = \delta_{m0}\sqrt{(2l+1)/4\pi}$. The formula for the energy of interaction becomes

$$
E = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\zeta\, \mathrm{tr}\ln(1-A), \tag{6.10}
$$

where the matrix

$$
A_{lm,l'm'} = \delta_{m,m'} \sum_{l''} B_{ll''m}(a) B_{l''l'm}(b) \tag{6.11}
$$

is given in terms of the quantities

$$
\begin{aligned}
B_{ll'm}(a) &= \frac{\sqrt{\pi}}{\sqrt{2\zeta R}} \mathrm{i}^{-l+l'} \sqrt{(2l+1)(2l'+1)} \sum_{l''} (2l''+1) \\
&\quad \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \frac{K_{l''+1/2}(\zeta R)\lambda_1 a I^2_{l'+1/2}(\zeta a)}{1+\lambda_1 a I_{l'+1/2}(\zeta a) K_{l'+1/2}(\zeta a)}.
\end{aligned} \tag{6.12}
$$

Note that the phase always cancels in the trace in equation (6.10). For strong coupling, this result reduces to that found by Bulgac, Wirzba *et al* in [12, 14] for Dirichlet spheres, and recently generalized by Emig *et al* in [2] for Robin boundary conditions; see also [44].

6.2. Weak coupling

For weak coupling, a major simplification results because of the orthogonality property,

$$
\sum_{m=-l}^{l} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l' & l''' \\ m & -m & 0 \end{pmatrix} = \delta_{l''l'''} \frac{1}{2l''+1}, \qquad l \leqslant l'. \tag{6.13}
$$

Then the formula for the energy of interaction between the two spheres is

$$
\begin{aligned}
E &= -\frac{\lambda_1 a \lambda_2 b}{4R} \int_0^\infty \frac{\mathrm{d}x}{x} \sum_{ll'l''} (2l+1)(2l'+1)(2l''+1) \\
&\quad \times \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}^2 K^2_{l''+1/2}(x) I^2_{l+1/2}(xa/R) I^2_{l'+1/2}(xb/R).
\end{aligned} \tag{6.14}
$$

There is no infrared divergence because for small x the product of Bessel functions goes like $x^{2(l+l'-l'')+1}$, and $l'' \leqslant l + l'$.

As with the cylinders, we expand the modified Bessel functions of the first kind in power series in $a/R, b/R < 1$. This expansion yields the infinite series

$$
E = -\frac{\lambda_1 a \lambda_2 b}{4\pi R} \frac{ab}{R^2} \sum_{n=0}^\infty \frac{1}{n+1} \sum_{m=0}^{n} D_{n,m} \left(\frac{a}{R}\right)^{2(n-m)} \left(\frac{b}{R}\right)^{2m}, \tag{6.15}
$$

where by inspecting the first several $D_{n,m}$ coefficients we can identify them as

$$
D_{n,m} = \frac{1}{2}\binom{2n+2}{2m+1}, \tag{6.16}
$$

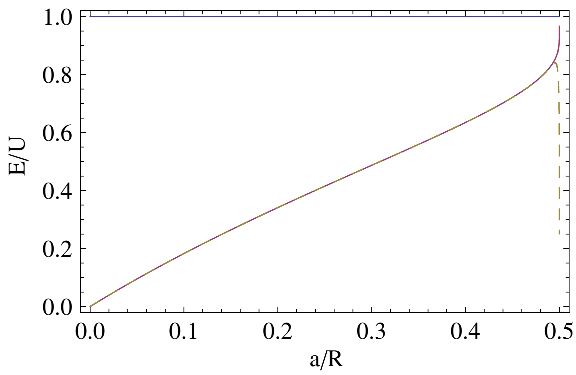

Figure 4. Plotted is the ratio of the exact interaction energy (6.17) of two weakly-coupled spheres to the proximity force approximation (6.18) as a function of the sphere radius a for $a = b$. Shown also by a dashed line is the power series expansion (6.15), truncated at $n = 100$, indicating that it is necessary to include very high powers to capture the proximity force limit.

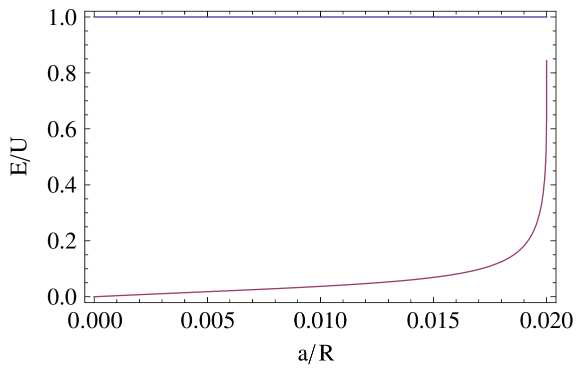

Figure 5. Plotted is the ratio of the exact interaction energy (6.17) of two weakly-coupled spheres to the proximity force approximation (6.18) as a function of the sphere radius a for $b/a = 49$.

and now we can immediately sum the expression (6.15) for the Casimir interaction energy to give the closed form

$$E = \frac{\lambda_1 a \lambda_2 b}{16\pi R} \ln\left(\frac{1 - \left(\frac{a+b}{R}\right)^2}{1 - \left(\frac{a-b}{R}\right)^2}\right). \tag{6.17}$$

Again, when $d = R - a - b \ll a, b$, the proximity force theorem (B.9) is reproduced:

$$U(d) \sim \frac{\lambda_1 \lambda_2 ab}{16\pi R} \ln(d/R), \qquad d \ll a, b. \tag{6.18}$$

However, as figures 4, 5 demonstrate, the approach is not very smooth, even for equal-sized spheres. The ratio of the energy to the PFA is

$$\frac{E}{U} = 1 + \frac{\ln[(1+\mu)^2/2\mu]}{\ln d/R}, \qquad d \ll a, b, \tag{6.19}$$

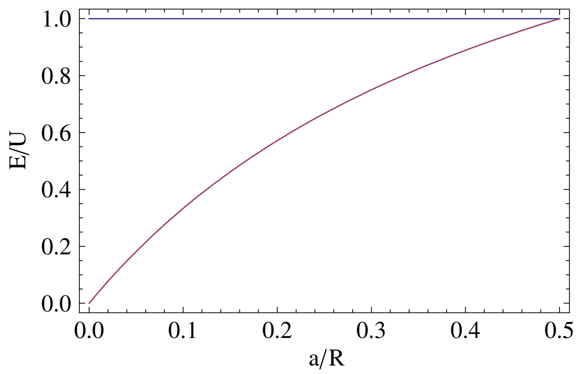

Figure 6. Plotted is the ratio of the exact interaction energy (6.22) of a weakly-coupled sphere above a Dirichlet plane to the proximity force approximation (6.23) as a function of the sphere radius a.

for $b/a = \mu$. Truncating the power series (6.15) at $n = 100$ would only begin to show the approach to the proximity force theorem limit. The error in using the PFA between spheres can be very substantial.

Again we will forego discussion of the strong-coupling (Dirichlet) limit here because of the extensive discussion already in the literature [2, 12, 14].

6.3. Exact result for interaction between plane and sphere

In just the way indicated above, we can obtain a closed-form result for the interaction energy between a weakly-coupled sphere and a Dirichlet plane. Using the simplification that

$$\sum_{m=-l}^{l}(-1)^m \begin{pmatrix} l & l & l' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} l & l & l' \\ 0 & 0 & 0 \end{pmatrix} = \delta_{l'0}, \tag{6.20}$$

we can write the interaction energy in the form

$$E = -\frac{\lambda a}{2\pi R}\int_0^\infty \mathrm{d}x \sum_{l=0}^{\infty}\sqrt{\frac{\pi}{2x}}(2l+1)K_{1/2}(x)I_{l+1/2}^2(x(a/R)). \tag{6.21}$$

Then in terms of $R/2$ as the distance between the center of the sphere and the plane, the exact interaction energy is

$$E = -\frac{\lambda}{2\pi}\left(\frac{a}{R}\right)^2 \frac{1}{1-(2a/R)^2}, \tag{6.22}$$

which as $a \to R/2$ reproduces the proximity force limit, contained in the (ambiguously defined) PFA formula

$$U = -\frac{\lambda}{8\pi}\frac{a}{d}. \tag{6.23}$$

The exact energy and this PFA approximation are compared in figure 6.

7. Comments and prognosis

Although the multiple scattering methods date back to the 1970s [5, 8] some remarkable new results have been obtained. Here we have given perhaps a simpler and more transparent derivation of the procedure than in [1, 2]. For example, because we have approached the problem from a general field theoretic viewpoint, we see that the 'translation matrix' introduced there is nothing other than the free Green's function. Our approach yields the general form first, and the multipole expansion as a derived consequence, not the other way around. We apply this multiple scattering method to obtain new results for the interaction between semitransparent cylinders and spheres, and we have analytically demonstrated the approach to the proximity force theorem. Most remarkably, we have derived explicit, very simple, closed-form expressions for the interaction between weakly coupled cylinders and between weakly coupled spheres, as well as between weakly-coupled cylinders or spheres and Dirichlet planes. These explicit results demonstrate the profound limitation of the proximity force approximation, which has been under serious criticism for some time [45, 46]. We hope that these developments will lead to improved conceptual understanding, and to better comparison with experiment, when they are extended to realistic materials.

Acknowledgments

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Appendix A. Derivation of vacuum energy formula

Following Schwinger [33] we start from the vacuum amplitude in terms of sources,

$$\langle 0_+|0_-\rangle^K = \mathrm{e}^{\mathrm{i}W[K]}, \qquad W[K] = \frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}x')K(x)G(x,x')K(x'). \tag{A.1}$$

Here G is the Green's function in the presence of some background potential. From this the effective field is

$$\phi(x) = \int(\mathrm{d}x')G(x,x')K(x'). \tag{A.2}$$

If the geometry of the region is altered slightly, as through moving one of the bounding surfaces, the vacuum amplitude is altered:

$$\delta W[K] = \frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}x')K(x)\delta G(x,x')K(x') = -\frac{1}{2}\int(\mathrm{d}x)(\mathrm{d}x')\phi(x)\delta G^{-1}(x,x')\phi(x'), \tag{A.3}$$

which uses the fact that

$$GG^{-1} = 1. \tag{A.4}$$

Upon comparison of equation (A.3) with the two-particle emission term in

$$\mathrm{e}^{\mathrm{i}W[K]} = \mathrm{e}^{\mathrm{i}\int(\mathrm{d}x)K(x)\phi(x)+\mathrm{i}\int(\mathrm{d}x)\mathcal{L}} = \cdots + \frac{1}{2}\left[\mathrm{i}\int(\mathrm{d}x)K(x)\phi(x)\right]^2, \tag{A.5}$$

we deduce from the coefficient of $\phi(x)\phi(x')$ that the effective two-particle source due to a geometry modification is

$$\mathrm{i}K(x)K(x')|_{\text{eff}} = -\delta G^{-1}(x,x'). \tag{A.6}$$

Thus the change in the generating functional is obtained by inserting equation (A.6) into equation (A.1),

$$\delta W = \frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')G(x,x')\delta G^{-1}(x',x) = -\frac{\mathrm{i}}{2}\int(\mathrm{d}x)(\mathrm{d}x')\delta G(x,x')G^{-1}(x',x), \tag{A.7}$$

which gives the change in the action under an alteration of the Green's function. From this, in matrix notation

$$\delta W = -\frac{\mathrm{i}}{2}\delta\,\mathrm{Tr}\ln G \quad\Rightarrow\quad E = \frac{\mathrm{i}}{2\tau}\mathrm{Tr}\ln G, \tag{A.8}$$

for a static configuration $W = -E\tau$, which is our starting point, equation (2.1).

There are of course many other derivations for this famous result. For example, one can derive it rather simply on the basis of Schwinger's quantum action principle. It may also be worth noting that it is formally equivalent to another familiar representation for the quantum vacuum energy

$$E = -\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\omega^2\,\mathrm{Tr}\,\mathcal{G}, \tag{A.9}$$

(for example, see [41]). Here, the Fourier transform of the Green's function appears

$$G(x,x') = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}\mathcal{G}(\mathbf{r},\mathbf{r}';\omega). \tag{A.10}$$

In terms of $\mathcal{G}$, equation (A.8) can be written in the form

$$E = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\mathrm{Tr}\ln\mathcal{G}. \tag{A.11}$$

Because

$$\mathcal{G}^{-1}\mathcal{G} = 1, \qquad \mathcal{G}^{-1} = -\omega^2 - \nabla^2 + V, \tag{A.12}$$

we see that when equation (A.11) is integrated by parts, and surface terms are ignored, we immediately recover equation (A.9).

Appendix B. Proximity force approximation

In this appendix we derive the proximity force approximation (PFA) for the energy of interaction between two semitransparent cylinders, or two semitransparent spheres, either in the strong or weak coupling regimes. This approximation, relating the force between nonplanar surfaces in terms of the forces between parallel plane surfaces, was first introduced in 1934 by Derjaguin in [47]. This approximation is only valid when the separation between the bodies is very small compared to their sizes. It is now well established that the approximation cannot be extended beyond that limit, and that 1% error occurs if the PFA is applied when the ratio of the separation to the radius of curvature of the bodies is of order 1%. Fortunately, current experiments have not exceeded this limit. This should change in the near future, which is one reason the new numerical calculations are of importance. In fact, we have found that in general the errors in using the PFA may be much larger than indicated above. We concur with Bordag that while the proximity force theorem is exact at zero separation, any approximation based on extrapolation away from that point is subject to uncontrollable errors.

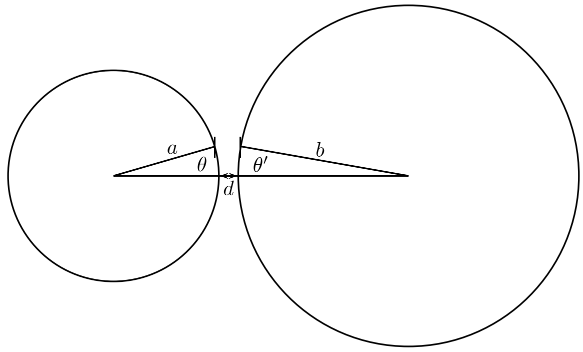

Figure B1. Geometry of two cylinders (or two spheres) with radii a and b, respectively, and distances between their centers of $R > a + b$. The proximity force approximation applies when the distance of closest approach $d = R - a - b \ll a, b$. The approximation consists in assuming that the interaction is dominated by the interaction of adjacent surface elements, as shown.

Consider first two parallel cylinders, of radius a and b, with their centers separated by a distance $R > a + b$. The distance of closest approach of the cylinders is $d = R - a - b$. The PFA consists of assuming that the energy between the two bodies is the sum of the energies between small parallel plane elements at the same height along the surfaces, that is, in polar coordinates at θ relative to the center of cylinder A and at θ' relative to the center of cylinder B, where as seen in figure B1,

$$a \sin\theta = b \sin\theta'. \tag{B.1}$$

Because d is much smaller than either a or b, only small values of θ actually contribute, and the energy of interaction $U(d)$ between the surface may be expressed in terms of the energy per unit area $\mathcal{E}(h)$ for the corresponding parallel plate problem, with separation distance h:

$$U(d) = \int a\,\mathrm{d}\theta\,\mathcal{E}[d + a(1 - \cos\theta) + b(1 - \cos\theta')]. \tag{B.2}$$

Here, for weak coupling (see equation (3.7)),

$$\mathcal{E}(h) = -\frac{\lambda_1\lambda_2}{32\pi^2 h}. \tag{B.3}$$

Because θ is small, the PFA energy per unit length is

$$U(d) = -\frac{\lambda_1\lambda_2}{32\pi^2}\frac{a}{d}\int_{-\pi}^{\pi}\mathrm{d}\theta\left[1 + \frac{a}{d}\left(1 + \frac{a}{b}\right)\frac{\theta^2}{2}\right]^{-1} = -\frac{\lambda_1\lambda_2}{32\pi}\sqrt{\frac{2ab}{R}}\frac{1}{d^{1/2}}. \tag{B.4}$$

To obtain the corresponding result for strong coupling, we merely replace $\mathcal{E}(h) = -\pi^2/(1440h^3)$, and a similar calculation yields

$$U(d) = -\frac{\pi^3}{3840}\sqrt{\frac{2ab}{R}}\frac{1}{d^{5/2}}, \qquad d \ll a, b. \tag{B.5}$$

It is easy to reproduce the result given by Bordag in [16] for a cylinder in front of a plane. For the strong coupling (Dirichlet) case we simply take result (B.5) and regard b as much larger than a, and obtain

$$U(d) = -\frac{\pi^3}{1920\sqrt{2}}\frac{a^{1/2}}{d^{5/2}}, \qquad d \ll a. \tag{B.6}$$

For a weakly coupled cylinder in front of a Dirichlet plane, we start from the corresponding interaction between two such planes, $\mathcal{E}(h) = -\lambda/(32\pi^2 h^2)$, which leads to

$$U(d) = -\frac{\lambda}{64\pi}\frac{(2a)^{1/2}}{d^{3/2}}. \tag{B.7}$$

For nearly touching spheres the calculation goes just the same way. The result, for strong coupling (Dirichlet boundary conditions), for the PFA energy is

$$U(d) = -\frac{\pi^3}{1440}\frac{ab}{R}\frac{1}{d^2}, \qquad d \ll a, b, \tag{B.8}$$

while in the weak-coupling limit there is sensitivity for large θ signifying a logarithmic divergence,

$$U(d) \sim \frac{\lambda_1\lambda_2 ab}{16\pi R}\ln(d/R), \qquad d \ll a, b. \tag{B.9}$$

For a weakly-coupled sphere in front of a Dirichlet plane, a PFA approximation is

$$U(d) = -\frac{\lambda}{16\pi}\frac{a}{d}. \tag{B.10}$$

Appendix C. Short distance limit

C.1. Cylinders

In this section of the appendix we want to discuss the short distance limit, for the case of weakly-coupled cylinders, where the closest distance between the cylinders is $R - a - b = d \ll a, b$, which should reduce to the proximity force approximation derived in appendix B. We will calculate the first correction to the PFA, and compare to the exact result found in section 5.4. In this limit, we replace the modified Bessel functions by their uniform asymptotic approximants, which in leading form yield

$$K^2_{m+m'}(x)I^2_m(xa/R)I^2_{m'}(xb/R) \sim \frac{1}{8\pi}\frac{1}{mm'(m+m')}tt_at_b\,\mathrm{e}^{-\chi}, \tag{C.1}$$

where

$$t = (1+z^2)^{-1/2}, \qquad t_a = \left(1+z_a^2\right)^{-1/2}, \qquad t_b = \left(1+z_b^2\right)^{-1/2}, \tag{C.2}$$

and

$$z = \frac{x}{m+m'}, \qquad z_a = \frac{xa/R}{m}, \qquad z_b = \frac{xb/R}{m'}. \tag{C.3}$$

The exponent here is

$$\chi = 2(m+m')\eta(z) - 2m\eta(z_a) - 2m'\eta(z_b), \tag{C.4}$$

where η is defined by equation (5.43). The reason that the force diverges as $a + b \to R$ is that χ vanishes here, for suitable values of m and m'. To make it systematic, let us rescale variables,

$$m = M\alpha, \qquad m' = M\beta, \tag{C.5}$$

and then when $b = R - a$, $\chi = 0$ when $\beta a = \alpha b$.

When $b = R - a - d$, with d small compared to the radius of either cylinder, we assume that the main contribution comes from the neighborhood of these values. So we define

$$\alpha = \frac{a}{R} + \hat{\alpha}, \qquad \beta = 1 - \frac{a}{R} + \hat{\beta}, \tag{C.6}$$

and we expand the exponent to first order in d and to second order in $\hat{\alpha}$ and $\hat{\beta} = -\hat{\alpha}$. (The latter constraint ensures that $\alpha + \beta = 1$.) The result is

$$\chi = \frac{2Md}{tR} + \frac{MtR^2\hat{\alpha}^2}{a(R-a)} + O(\hat{\alpha}^3, d^2). \tag{C.7}$$

Then

$$\begin{aligned}\mathfrak{E} &\sim -\frac{\lambda_1\lambda_2}{16\pi^2}\int_0^\infty \mathrm{d}z\, zt^3 \int_0^\infty \mathrm{d}M\, \mathrm{e}^{-2Md/tR}\int_{-\infty}^\infty \mathrm{d}\hat{\alpha}\, \mathrm{e}^{-Mt\hat{\alpha}^2R^2/[a(R-a)]} \\ &= -\frac{\lambda_1\lambda_2}{32\pi}\sqrt{\frac{2}{d}}\sqrt{\frac{a(R-a)}{R}} = U,\end{aligned} \tag{C.8}$$

which is exactly the result given by the proximity force theorem in appendix B, equation (B.4).

Now we calculate the correction to the PFA. We do this by keeping subleading terms in the uniform asymptotic approximation for the product of six Bessel function

$$\begin{aligned}K_{m+m'}^2(x)I_m^2(xa/R)I_{m'}^2(xb/R) &\sim \frac{1}{8\pi mm'}\frac{tt_at_b}{m+m'} \\ &\times \left(1 - \frac{u_1(t)}{m+m'}\right)^2\left(1 + \frac{u_1(t_a)}{m}\right)^2\left(1 + \frac{u_1(t_b)}{m'}\right)^2 \mathrm{e}^{-\chi},\end{aligned} \tag{C.9}$$

where $t = t(z)$ with $z = x/(m+m')$, $z_a = xa/m$, $z_b = xb/m'$,

$$u_1(t) = \frac{3t - 5t^3}{24}, \tag{C.10}$$

and χ is given by equation (C.4). Now when we expand χ we must go out to the order $\hat{\alpha}^4$, d^2, and $\hat{\alpha}^2 d$. The result is

$$\begin{aligned}\mathrm{e}^{-\chi} \sim \mathrm{e}^{-2Md/tR}\mathrm{e}^{-M\hat{\alpha}^2tR^2/(a(R-a))}\Bigg[&1 - \frac{d^2Mt}{R(R-a)} + \frac{2\hat{\alpha}dMt}{R-a} \\ &- \frac{\hat{\alpha}^2dMt(1-t^2)R}{(R-a)^2} + \frac{\hat{\alpha}^3Mt^3(R-2a)R^3}{3a^2(R-a)^2} + \frac{\hat{\alpha}^4Mt^3(1-3t^2)(R^2-3aR+3a^2)R^4}{12a^3(R-a)^3} \\ &+ \frac{2M^2\hat{\alpha}^2d^2t^2}{(R-a)^2} + \frac{M^2\hat{\alpha}^6t^6(R-2a)^2R^6}{18a^4(R-a)^4} + \frac{2}{3}\frac{M^2\hat{\alpha}^4t^4dR^3}{a^2(R-a)^3}(R-2a)\Bigg].\end{aligned} \tag{C.11}$$

As above, we replace

$$m = M\frac{a}{R}\left(1 + \hat{\alpha}\frac{R}{a}\right), \qquad m' = M\left(1 - \frac{a}{R}\right)\left(1 - \hat{\alpha}\frac{R}{R-a}\right). \tag{C.12}$$

We expand t_a and t_b in the prefactor using

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -zt^3, \qquad \frac{\mathrm{d}^2t}{\mathrm{d}z^2} = 2t^3 - 3t^5. \tag{C.13}$$

The PFA is obtained by using the integrals

$$\int_{-\infty}^\infty \mathrm{d}\hat{\alpha}\, \mathrm{e}^{-\hat{\alpha}^2\gamma} = \sqrt{\frac{\pi}{\gamma}}, \qquad \gamma = MtR^2/a(R-a), \tag{C.14a}$$

$$\int_0^\infty \frac{\mathrm{d}M}{\sqrt{M}}\,\mathrm{e}^{-2Md/t} = \Gamma\left(\frac{1}{2}\right)\left(\frac{2d}{t}\right)^{-1/2}, \tag{C.14b}$$

and so, from the expansion we can obtain the result of the integrals over $\hat{\alpha}$ and M by the algebraic substitutions

$$\frac{1}{M} \to -\frac{4d}{Rt}, \qquad M \to \frac{tR}{4d}, \tag{C.15a}$$

$$\hat{\alpha}^2 \to -\frac{2a(R-a)d}{R^3t^2}, \qquad M\hat{\alpha}^2 \to \frac{1}{2}\frac{a(R-a)}{R^2t}, \qquad M\hat{\alpha}^4 \to -\frac{3a^2(R-a)^2d}{R^5t^3}, \tag{C.15b}$$

$$M^2\hat{\alpha}^2 \to \frac{a(R-a)}{8Rd}, \qquad M^2\hat{\alpha}^4 \to \frac{3}{4}\frac{a^2(r-a)^2}{R^4t^2}, \qquad M^2\hat{\alpha}^6 \to -\frac{15}{2}\frac{a^3(R-a)^3d}{R^7t^4}. \tag{C.15c}$$

The result is the following correction factor to the PFA in the form given in equation (C.8):

$$\frac{\mathfrak{E}}{U} = 1 - \frac{R^2+aR+a^2}{4a(R-a)}\frac{d}{R}. \tag{C.16}$$

Although this looks slightly different from equation (5.37), it agrees with the latter when the PFA formula (5.36) is expressed in terms of the form given in equation (C.8), that is, writing $d = R - a - b$.

C.2. Spheres

Here we see how the proximity force limit is achieved for weakly-coupled spheres. Again, the strategy is to replace the modified Bessel functions by their leading uniform asymptotic approximants. The only new element is the appearance of the 3-j symbol. Because now only $m = 0$ appears, there is a very simple approximant for the latter [48–50]:

$$\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \sim \sqrt{\frac{\pi}{2}}\frac{\cos\frac{\pi}{2}(l+l'+l'')}{[(l+l'+l'')(l+l'-l'')(l-l'+l'')(-l+l'+l'')]^{1/4}}. \tag{C.17}$$

For more on the asymptotics of Clebsch–Gordon coefficients see [24]. This result is quite accurate, being within 1% of the true value of the Wigner coefficient for l's of order 100 (except very near the boundaries of the triangular region, where the approximant diverges weakly). Otherwise, the procedure is rather routine. Letting $\nu = l + 1/2$, and similarly for the primed quantities, we expand the exponent resulting from the uniform asymptotic expansion about the critical point, with

$$\nu = N\left(\frac{a}{R} + \hat{\alpha}\right), \qquad \nu' = N\left(1 - \frac{a}{R} + \hat{\alpha}'\right), \qquad \nu'' = N(1 + \alpha''), \tag{C.18}$$

with the constraint $\hat{\alpha} + \hat{\alpha}' + \hat{\alpha}'' = 0$. Replacing the sums over angular momenta by integrals, and changing variables:

$$\int \mathrm{d}\nu\,\mathrm{d}\nu'\,\mathrm{d}\nu'' = 2\int_0^\infty \mathrm{d}N\,N^2\int_0^\infty \mathrm{d}(\hat{\alpha}+\hat{\alpha}')\int_{-\infty}^\infty \frac{\mathrm{d}(\hat{\alpha}-\hat{\alpha}')}{2}, \tag{C.19}$$

which reflects the restriction emerging from the triangular relation of the Wigner coefficients, $\hat{\alpha} + \hat{\alpha}' > 0$, we find for the approximant to the energy when the two spheres are nearly touching:

$$E \sim -\frac{\lambda_1 a \lambda_2 b}{4R}\frac{2}{\pi}\int_0^\infty \frac{\mathrm{d}x}{x}\int_0^\infty \mathrm{d}N\, N^2 t^3\, \mathrm{e}^{-2Nd/Rt}\frac{1}{4\pi N^2}$$
$$\times\left[\frac{R^2}{a(1-a)}\right]^{1/2}\int_0^\infty \frac{\mathrm{d}(\hat\alpha+\hat\alpha')}{(\hat\alpha+\hat\alpha')^{1/2}}\,\mathrm{e}^{-4N(\hat\alpha+\hat\alpha')/t}$$
$$\times\int_{-\infty}^\infty \mathrm{d}\left(\frac{\hat\alpha-\hat\alpha'}{2}\right)\mathrm{e}^{-NtR^2(\hat\alpha-\hat\alpha')^2/4a(R-a)}$$
$$\sim \frac{\lambda_1\lambda_2 ab}{16\pi R}\ln d, \qquad d = R-a-b \ll a, b, \tag{C.20}$$

which is exactly the PFA result (B.9).

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